MATH 2700 CURVE ANALYSIS: DEFINITIONS, DERIVATIONS, AND PROOFS (OH MY!)

- $\vec{r}(t)$ is the **position** function; $\vec{r}(t)$ tells where you are at time t.
- $\vec{v}(t) = \vec{r}\,'(t)$ is the **velocity** function; $\vec{v}(t)$ tells you how fast and in which direction you're moving.
- $\|\vec{v}(t)\|$ is the **speed**; $\|\vec{v}(t)\|$ tells you how fast you're going.

  NOTE: Think of $\|\vec{v(t)}\|$ as the 'speedometer' function.
- $\hat{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}$ is the **principal unit tangent vector**; $\hat{T}(t)$ tells you the direction you're going.

  NOTE: For $\hat{T}(t)$ to exist, $\|\vec{v(t)}\| \neq 0$. In this case, $\vec{v}(t) = \|\vec{v}(t)\|\,\hat{T}(t)$.
- $\vec{r}(t)$ is **smooth** if $\vec{r}\,'(t) = \vec{v}(t) \neq \vec{0}$ or, equivalently, $\|\vec{r}\,'(t)\| = \|\vec{v}(t)\| \neq 0$.
- $s(t) = \displaystyle\int_a^t \|\vec{v}(u)\|\,du$ is the **arc length parameter**. It tells you how far you've travelled over $[a, t]$.

  NOTE: Think of $s(t)$ as the 'odometer' function and note that: $\dfrac{ds}{dt} = \|\vec{v}(t)\|$.
- $\hat{N}(t) = \dfrac{\hat{T}\,'(t)}{\|\hat{T}\,'(t)\|}$ is the **principal unit normal vector**; $\hat{N}(t)$ tells you the direction you're turning.

  NOTE: For $\hat{N}(t)$ to exist, $\|\hat{T}\,'(t)\| \neq 0$; i.e., $\hat{T}(t)$ needs to be smooth for $\hat{N}(t)$ to exist.
- $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$ is the **principal unit binormal vector**.

  NOTE: $\hat{B}(t)$ is orthogonal to both $\hat{T}(t)$ and $\hat{N}(t)$ and $\|\hat{B}(t)\| = 1$.
- The **Frenet Frame** or **TNB-frame** is 3-D coordinate system determined by $\hat{T}(t)$, $\hat{N}(t)$, and $\hat{B}(t)$.
  - The **osculating plane** is the plane determined by $\hat{T}(t)$ and $\hat{N}(t)$ with normal vector $\hat{B}(t)$.
  - The **normal plane** is the plane determined by $\hat{N}(t)$ and $\hat{B}(t)$ with normal vector $\hat{T}(t)$.
  - The **rectifying plane** is the plane determined by $\hat{T}(t)$ and $\hat{B}(t)$ with normal vector $\hat{N}(t)$.
- $\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t)$ is the **acceleration function**.

  NOTE: $\vec{F}(t) = m\vec{a}(t)$, so $\vec{a}(t)$ is a scalar multiple of the force which keeps you on the path $\vec{r}(t)$.
- $\kappa(s) = \left\|\dfrac{d\hat{T}}{ds}\right\| = \left\|\hat{T}\,'(s)\right\| = \dfrac{d\hat{T}}{ds} \cdot \hat{N}$ is the **curvature** function.

  NOTE: $\kappa(s)$ measures how fast you're turning as a function of how far you've traveled, or, alternatively, the tendency to 'twist' out of the normal plane.
- $\tau(s) = -\dfrac{d\hat{B}}{ds} \cdot \hat{N}(s)$ is the **torsion** function.

  NOTE: $\tau(s)$ measures the rate at which the motion is 'twisting' out of the osculating plane.

CURVE ANALYSIS FORMULA SUMMARY SHEET

- **POSITION:** $\vec{r}$
- **VELOCITY:** $\vec{v} = \vec{r}\,'$
- **ACCELERATION:** $\vec{a} = \vec{v}\,'$
- **PRINCIPAL UNIT TANGENT VECTOR** $\hat{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$
- **PRINCIPAL UNIT NORMAL VECTOR:** $\hat{N} = \dfrac{\hat{T}\,'}{\|\hat{T}\,'\|}$
- **COMPONENTS OF ACCELERATION:** $\vec{a} = a_T\hat{T} + a_N\hat{N}$ where $a_T = \dfrac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|}$ and $a_N = \dfrac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|}$
- **PRINCIPAL BINORMAL VECTOR:** $\hat{B} = \hat{T}\times\hat{N} = \dfrac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}$
- **CURVATURE:** $\kappa = \left\|\dfrac{d\hat{T}}{ds}\right\| = \dfrac{d\hat{T}}{ds}\cdot\hat{N} = \dfrac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|^3}$

  NOTE: $\dfrac{d\hat{T}}{ds} = \kappa\hat{N}$
- **TORSION:** $\tau = -\dfrac{d\hat{B}}{ds}\cdot\hat{N} = \dfrac{(\vec{v}\times\vec{a})\cdot\vec{a}\,'}{\|\vec{v}\times\vec{a}\|^2}$.

  NOTE: $\dfrac{d\hat{B}}{ds} = -\tau\hat{N}$
- **FRENET - SERRET EQUATIONS:**

  From $\dfrac{d\hat{T}}{ds} = \kappa\hat{N}$ and $\dfrac{d\hat{B}}{ds} = -\tau\hat{N}$, we can differentiate $\hat{N} = \hat{B}\times\hat{T}$, to get:

$$\begin{cases} \dfrac{d\hat{T}}{ds} &= \kappa\,\hat{N} \\[2ex] \dfrac{d\hat{N}}{ds} &= -\kappa\,\hat{T} + \tau\,\hat{B} \\[2ex] \dfrac{d\hat{B}}{ds} &= -\tau\,\hat{N} \end{cases}$$